Electromagnetic potential: Magnetic moment of the electron

If A and ϕ are electromagnetic vectar and scalar potentials, then

$$p \to p - \frac{eA}{c} \quad \text{or} \quad cP \to cP - eA$$

$$E \to E - e\phi$$

Here e represent charge on the particle.

So that the Dirac equation

$$(E - c\vec{\alpha} \cdot P - \beta mc^2)\psi = 0$$

take the form

$$[E - e\phi - \vec{\alpha}(cP - eA) - \beta mc^2]\psi = 0 \quad \text{— (1)}$$

Here E and P are

$$E \to i\hbar \frac{\partial}{\partial t}, \quad P = -i\hbar\nabla$$

Multiplying equation (1) by

$$[E - e\phi + \vec{\alpha}(cP - eA) + \beta mc^2]$$

or $[E - e\phi + \vec{\alpha} \cdot (cP - eA) + \beta mc^2] \times$

$$[E - e\phi + \vec{\alpha} \cdot (cP - eA) - \beta mc^2]\psi = 0$$

$$[(E - e\phi)^2 - \{\vec{\alpha} \cdot (c\vec{P} - eA)\}^2 - (E - e\phi)\vec{\alpha} \cdot (cP - eA)$$
$$+ \vec{\alpha} \cdot (cP - eA)(E - e\phi)]\psi = 0 \quad \text{— (2)}$$

$$\beta^2 = 1$$

We know that

$$(\vec{\alpha} \cdot B)(\vec{\alpha} \cdot C) = B \cdot C + i\vec{\sigma}' \cdot (B \times C) \quad \text{— (3)}$$

replace B and c both by $(cP - eA)$ then.

$$[\vec{\alpha}.(cP-eA)]^2 = (cP-eA)^2 + i\vec{\sigma}'.(cP-eA)\times(cP-eA) \quad \text{---(4)}$$

But $(cP-eA)\times(cP-eA) = -ce(A\times P + P\times A)$ — (5)

$$[(A\times P)+(P\times A)]\psi = A\times P\psi + P\times A\psi$$

$$= A\times\frac{\hbar}{i}\nabla\psi + \frac{\hbar}{i}\nabla\times(A\psi)$$

$$= A\times\frac{\hbar}{i}\nabla\psi + \frac{\hbar}{i}(\psi\nabla\times A + \nabla\psi\times A)$$

$$\boxed{\nabla\times(\psi A) = \psi\nabla\times A + \nabla\psi\times A}$$

$$= A\times\frac{\hbar}{i}\nabla\psi + \frac{\hbar}{i}(\psi\nabla\times A + A\times\nabla\psi) = \left[\frac{\hbar}{i}\nabla\times A\right]\psi$$

$$A\times P + P\times A = \frac{\hbar}{i}\nabla\times A = i\hbar\nabla\times A \quad \text{--- (6)}$$

using this result equation (5) become.

$$(cP-eA)\times(cP-eA) = -ce(-i\hbar\nabla\times A) = iec\nabla\times A$$

Putting $\nabla\times A = B$

$$(cP-eA)\times(cP-eA) = iec\hbar B \quad \text{--- (7)}$$

Hence equation 4

$$[\vec{\alpha}\cdot(cP-eA)]^2 = (cP-eA)^2 + i\vec{\sigma}'.iec\hbar B$$

$$= (cP-eA)^2 - e\hbar c\vec{\sigma}'.B \quad \text{--- (8)}$$

The last two operators in equation (2) can be simplified as follows.

$$-(E-e\phi)\vec{\alpha}.(cP-eA) + \vec{\alpha}.(cP-eA)(E-e\phi)$$

$$= e\vec{\alpha}.(EA-AE) + ce\vec{\alpha}.(\phi-P\phi$$

$$= e\vec{\alpha} \cdot i\hbar \frac{\partial A}{\partial t} + ce\vec{\alpha} \cdot i\hbar \nabla \phi$$

$E = i\hbar \frac{\partial}{\partial t}$ and $p = \frac{\hbar}{i} \nabla$ and then ~~EA-AE~~

$$EA - AE = i\hbar \frac{\partial A}{\partial t}, \quad \phi P - P\phi = i\hbar \nabla \phi$$

$$= ie\hbar c \vec{\alpha} \cdot \left[-\frac{1}{c} \frac{\partial A}{\partial t} - \nabla \phi \right]$$

$$= ie\hbar c \vec{\alpha} \cdot E \qquad — (9) \quad \left[E = -\frac{1}{c} \frac{\partial A}{\partial t} - \nabla \phi \right]$$

Putting the values from 8 and 9 in equation 2

$$\left[(E - e\phi)^2 - [(cP - eA)^2 - e\hbar c \vec{\sigma}' \cdot B] - m^2c^4 - ie\hbar c \vec{\alpha} \cdot E \right] \psi = 0$$

$$\left[(E - e\phi)^2 - (cP - eA)^2 - m^2c^4 + e\hbar c \vec{\sigma}' \cdot B - ie\hbar c \vec{\alpha} \cdot E \right] \psi = 0 \qquad (10)$$

The last two term may be understood by taking non-relativistic limit of the entire equation.

In the non-relativistic limit

$$E \to E' + mc^2 \text{ and } E' \ll mc^2 ; \quad e\phi \ll mc^2$$

$$(E - e\phi)^2 = (E' + mc^2 - e\phi)^2$$

$$= m^2c^4 \left[1 + \frac{E' - e\phi}{mc^2} \right]^2$$

$$= m^2c^4 \left[1 + \frac{2(E' - e\phi)}{mc^2} \right]$$

$$= mc^4 + 2(E' - e\phi) mc^2$$

$$(E - e\phi)^2 - m^2c^4 = 2mc^2 (E' - e\phi) \qquad — (11)$$

Substituting values from (11) to (10)

$$(2mc^2(E'-e\phi)-(cP-eA)^2+e\hbar c\vec{\sigma}'.B-ie\hbar\alpha.E)\psi=0$$

$$E'\psi=\left[\frac{1}{2m}(P-\frac{eA}{c})+e\phi-\frac{e\hbar}{2mc}\vec{\sigma}'\cdot B+\frac{ie\hbar}{2mc}\vec{\alpha}\cdot E\right]\psi \quad (12)$$

for a free particle $E\psi=\frac{P^2}{2m}\psi$

In E.M. field it takes the form.

$$(E-e\phi)\psi=\left[\frac{1}{2m}(P-\frac{eA}{c})^2\right]\psi$$

$$E\psi=\left[e\phi+\frac{1}{2m}(P-\frac{eA}{c})^2\right]\psi \quad \text{— (13)}$$

comparing (12) & (13) $E'=E\to i\hbar\frac{\partial}{\partial t}$

equ-(12) containing two extra term E & B, these term represent Electric & magnetic energy

We know magnetic energy $=-\mu\vec{B}$ when μ is magnetic moment from (12) third term

$\vec{\mu}=\frac{e\hbar}{2mc}\vec{\sigma}'$ magnetic dipole moment

Similarly electric dipole moment $=-\frac{ie\hbar\vec{\alpha}}{2mc}$

the term E in (12) is of order in practical $\frac{v^2}{c}$ times the $e\phi$ term so it may be neglected in the non-relativistic term.

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